Conclusions

Parametric calculations of supersonic flowfields around a series of rhombic delta wings were carried out to construct a correlation law for predicting the shock angle in the plane of symmetry of the wing. The computed shock angles were correlated by a parameter comprising Mach number, the theoretical two-dimensional oblique shock angle, and the angle between the plane of symmetry and the face of the rhombic wing. By approximating the correlated data using a polynomial fit, a correlation law of the shock angle with supersonic Mach number and wing geometry has been constructed.

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Eigenvector Derivatives for Doubly Repeated Eigenvalues

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Introduction

THE calculation of eigenvector derivatives with respect to a design variable has been discussed by various researchers.[1–3] If the eigensystem has distinct eigenvalues, these methods compute the eigenvector derivatives analytically. If, however, repeated eigenvalues occur in the system, the eigenvector derivatives with repeated eigenvalues can not be obtained by these methods. Ojalvo,[4] Dailey,[5] and Mills-Curran[6,7] addressed this problem and developed some approaches to compute the eigenvector derivatives.

Mills-Curran's method[6,7] was proved to be the most general one in solving this problem. In his approach the eigenvector derivatives with repeated eigenvalues are assumed to be equal to an unknown vector plus a linear combination of the eigenvectors with repeated eigenvalues. The unknown vector is determined by solving a set of linear equations. The coefficient matrix of these linear equations is obtained by deleting some rows and columns of the original singular eigenmatrix. A procedure was provided in his paper to find the nonsingular submatrix from the singular eigenmatrix with a higher order deficiency. For complex structures the procedure may become tedious.

It is found in this research that for a special case there is no need to employ Mills-Curran's method to compute the eigenvector derivatives with repeated eigenvalues. The eigenvector derivatives with repeated eigenvalues can still be computed using Fox and Kapoor's approach.[1] The special case is defined as follows. The design variable change affects the stiffness in only one direction and the mass matrix is not affected. If the system has identical stiffness in two directions, then the occurrence of doubly repeated eigenvalues can be expected. Owing to the dynamic characteristics of the structure, one of the eigenvalue sensitivities for the doubly repeated eigenvalues is zero.

Many structures have these characteristics.[8] For example, a cantilever beam has identical moments of inertia in the y and z directions. The occurrence of repeated eigenvalues is expected due to the existence of two identical perpendicular bending modes. If the change of any of the moments of inertia does not affect the other moment of inertia and the mass matrix, then the structure fulfills the definition of the problem.

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Theoretical Derivation

Let

$$[F_i] = [K] - \lambda_i[M]; \qquad i = 1, 2 \tag{1}$$

where $[K]$ is the stiffness matrix, $[M]$ the mass matrix, and λ_i the repeated eigenvalue.

The eigenproblem is expressed as

$$[F_i]\{\phi_i\} = 0; \qquad i = 1, 2 \tag{2}$$

where $\{\phi_i\}$ are the orthonormal eigenvectors associated with the repeated eigenvalue.

It is well known that the eigenvectors associated with the repeated eigenvalues are not unique. They can be the linear combination of the eigenvectors associated with the repeated eigenvalues,

$$\{\bar{\phi}_i\} = [\Phi]\{a_i\}; \qquad i = 1, 2 \tag{3}$$

where $[\Phi]$ contains the two eigenvectors associated with the repeated eigenvalues. In Eq. (3), $\{\bar{\phi}_i\}$ is the eigenvector based on a specific linear combination.

The weighting coefficients in $\{a_i\}$ can be obtained by solving the following subeigenproblem:

$$\left([\Phi]^T([K]' - \lambda_i[M]')[\Phi] - \lambda_i'[I]\right)\{a_i\} = 0; \qquad i = 1, 2 \tag{4}$$

where $[K]'$ and $[M]'$ are the derivatives of the stiffness and the mass matrices with respect to the design variable, respectively; λ_i' is the eigenvalue derivative that is also the eigenvalue of the subeigenproblem; and $[I]$ is the unit matrix.

If the eigenvalues of this subeigenproblem are distinct, then the eigenvectors of the original system are uniquely determined by Eq. (3). The eigenproblem (2) now becomes

$$[F_i]\{\bar{\phi}_i\} = 0; \qquad i = 1, 2 \tag{5}$$

Taking the derivatives of Eq. (5) with respect to a design variable yields

$$[F_i]\{\bar{\phi}_i\}' = -[F_i]'\{\bar{\phi}_i\}; \qquad i = 1, 2 \tag{6}$$

where $\{\bar{\phi}_i\}'$ and $[F_i]'$ are the derivatives of $\{\bar{\phi}_i\}$ and $[F_i]$, respectively.

Assume

$$\{\bar{\phi}_i\}' = \sum_{l=1}^{n} c_{li}\{\phi_l\}; \qquad i = 1, 2 \tag{7}$$

where n is the number of the degrees of freedom of the system. Substituting Eq. (7) into Eq. (6) and premultiplying Eq. (6) by $\{\phi_j\}^T$ gives

$$\{\phi_j\}^T[F_i]c_{ji}\{\phi_j\} = -\{\phi_j\}^T[F_i]'\{\bar{\phi}_i\} \tag{8}$$

or

$$c_{ji}(\lambda_j - \lambda_i) = -\{\phi_j\}^T[F_i]'\{\bar{\phi}_i\} \tag{9}$$

or

$$c_{ji} = -\frac{\{\phi_j\}^T[F_i]'\{\bar{\phi}_i\}}{\lambda_j - \lambda_i} \tag{10}$$

Table 1 Eigenvalues and eigenvectors of the system

Mode no.	1	2	3	4	5	6	7	8
λ_i	0.197	0.197	0.484	0.484	22.75	22.75	103.65	103.65
	−0.0970	0	0.0263	0	0.1262	0	0.0674	0
	0	−0.0970	0	−0.0263	0	−0.1262	0	0.0674
	0	−0.0129	0	−0.0102	0	−0.0926	0	0.0770
$\{\phi_i\}$	0.0129	0	−0.0102	0	−0.0926	0	−0.0770	0
	0.0030	0	−0.0687	0	0.0517	0	−0.0888	0
	0	0.0030	0	0.0687	0	−0.0517	0	−0.0888
	0	−0.0022	0	−0.0094	0	0.0251	0	0.0816
	0.0022	0	−0.0094	0	0.0251	0	−0.0816	0

Equation (10) is valid when j is not related to the repeated eigenvalues. The two weighting coefficients associated with the eigenvectors with repeated eigenvalues can be proved to be zero as follows. Assume the index i represents the first of the two repeated modes and the second repeated mode is designated by $i+1$.

For the two eigenvectors with repeated eigenvalues, the orthogonality condition requires that

$$\{\bar{\phi}_i\}^T[M]\{\bar{\phi}_i\} = 1 \tag{11}$$

and

$$\{\bar{\phi}_i\}^T[M]\{\bar{\phi}_{i+1}\} = 0 \tag{12}$$

Taking the derivatives of Eq. (11) with respect to the design variable results in

$$2\{\bar{\phi}_i\}^T[M]\{\bar{\phi}_i\}' = -\{\bar{\phi}_i\}^T[M]'\{\bar{\phi}_i\} \tag{13}$$

Substituting Eq. (7) into Eq. (13) yields

$$c_{ii} = -\tfrac{1}{2}\{\bar{\phi}_i\}^T[M]'\{\bar{\phi}_i\} \tag{14}$$

Since $[M]$ is not affected by the change of the design variable, $[M]' = 0$. Thus,

$$c_{ii} = 0 \tag{15}$$

Taking the derivatives of Eq. (12) with respect to the design variable yields

$$\{\bar{\phi}_i\}'^T[M]\{\bar{\phi}_{i+1}\} + \{\bar{\phi}_i\}^T[M]'\{\bar{\phi}_{i+1}\} + \{\bar{\phi}_i\}^T[M]\{\bar{\phi}_{i+1}\}' = 0 \tag{16}$$

Because the design variable change affects the structural stiffness in only one direction and the mass matrix remains unchanged, one of the eigenvectors associated with the repeated eigenvalues will change with the design variable and the other one will remain unchanged. Assume that $\{\bar{\phi}_{i+1}\}$ remains unchanged, hence, $\{\bar{\phi}_{i+1}\}' = 0$. Equation (16) now becomes

$$\{\bar{\phi}_{i+1}\}^T[M]\{\bar{\phi}_i\}' = 0 \tag{17}$$

Substituting Eq. (7) into Eq. (17) results in

$$c_{i+1,i} = 0 \tag{18}$$

The other method proposed in Fox and Kapoor's[1] paper is to augment the $[F_i]$ matrix with the row vector $\{\bar{\phi}_i\}^T[M]$ to make the $[F_i]$ matrix with full rank solve for $\{\bar{\phi}_i\}'$ by Eq. (6). A similar approach can be developed by augmenting Eq. (6) by Eqs. (13) and (17) as

$$[A]\{\bar{\phi}_i\}' = \{b\} \tag{19}$$

where

$$[A] = \begin{bmatrix} [F_i] \\ \text{--------} \\ \{\bar{\phi}_i\}^T[M] \\ \text{--------} \\ \{\bar{\phi}_{i+1}\}^T[M] \end{bmatrix}_{(n+2)\times n} \tag{20}$$

Table 2 Eigenvector derivatives ($\times 10^{-3}$)

$\{\phi_1\}'$	$\{\phi_2\}'$	$\{\phi_3\}'$	$\{\phi_4\}'$	$\{\phi_5\}'$	$\{\phi_6\}'$	$\{\phi_7\}'$	$\{\phi_8\}'$
0.1463	0	−0.1464	0	−1.3934	0	2.8770	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
−0.0588	0	0.0966	0	1.8252	0	−2.2163	0
0.2986	0	−0.0220	0	2.0826	0	1.2406	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0.2095	0	−0.0020	0	1.9606	0	0.6096	0

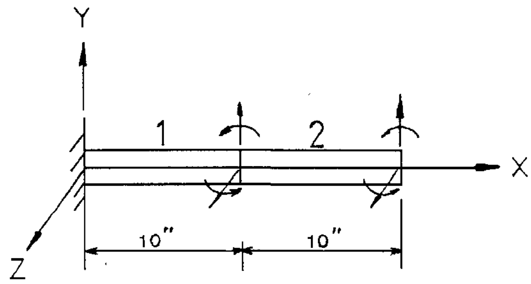

Fig. 1 Cantilever beam: A_1 = 20 in.^{2}, A_2 = 10 in.^{2}, I_1 = 33.3333 in.^{4}, I_2 = 8.3333 in.^{4}, E = 1000 psi, ρ = 0.001 lb-s^2 in.^{4}

and

$$\{b\} = \begin{Bmatrix} -[F_i]'\{\bar{\phi}_i\} \\ \text{--------} \\ 0 \\ \text{--------} \\ 0 \end{Bmatrix}_{(n+2)\times 1} \tag{21}$$

Because the system is assumed to have doubly repeated eigenvalues, the rank of matrix $[F_i]$ is $n-2$. The augmentation of two linearly independent rows to $[F_i]$ will result in a matrix with rank n. Therefore, Eq. (19) is not overdetermined, and the solution exists. Premultiplying Eq. (19) by $[A]^T$ results in an $n \times n$ square matrix with full rank. The eigenvector derivatives can be obtained by solving the following equation:

$$[A]^T[A]\{\bar{\phi}_i\}' = [A]^T\{b\} \tag{22}$$

Numerical Example

The numerical example illustrated is a two element cantilever beam shown in Fig. 1. Four sets of repeated eigenvalues occur. The moment of inertia I_{zz} of the second element is chosen as the design variable. Table 1 shows the eigenvalues and the eigenvectors of the structure. Table 2 shows the eigenvector derivatives with respect to I_{zz} of the second element by the present approach. The data in Table 2 are compared with the results using Mills-Curran's approach.[6,7] They are exactly the same.

Concluding Remarks

An easier way is found to calculate the eigenvector derivatives with repeated eigenvalues for some structures. The method, in general, follows Fox and Kapoor's[1] approach with some modifications. The results obtained are exactly the same as those by other methods.

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Topology Optimization with Superelements

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Introduction

PREVIOUS research on topology optimization focused on material microstructure modeling and efficient optimization techniques.[1–5] In general, the entire structure is analyzed and subsequently optimized. However, in a real design environment, it often occurs that only a small part of a large structure is allowed to change repeatedly and the rest is kept unchanged. Significant savings can be achieved if the analysis code takes this into account. The superelement or substructure formulation used in the finite element method is most suitable for this purpose.

Optimal design of large structures with substructuring or superelements was discussed in the literature.[6–8] Botkin and Yang[9] identified that tremendous savings can be achieved in both finite element analysis and design sensitivity analysis for three-dimensional shape optimization. In topology optimization, the computational advantages are similar to those in shape optimization. Tenek and Hagiwara[10] used a substructure method for obtaining optimum vehicle body panel topologies. Two iteration loops were proposed. The outer loop was the analysis loop that was concerned with the displacement field of the full vehicle finite element analysis. The inner loop was the topology optimization loop for the target panel that was removed from the vehicle structure. Since the target panel was totally disconnected from the full body during the topology optimization process, the convergence was not as good as that for considering full structure.

In this research, the superelement method was employed for finite element analysis. The target panel was modeled as the residual structure and the rest as one big substructure. Unlike the approach of Tenek and Hagiwara,[10] only one optimization loop is required and no analysis approximation is made during optimization iterations. This results in a better convergence characteristic and a more efficient optimization process as opposed to that for the full structure analysis.

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Superelement Method

The superelement method first divides the structure into a number of smaller substructures. For each substructure the nodes that are common to adjoining substructures are called boundary nodes. The degrees of freedom of these nodes are called boundary degrees of freedom. The nodes that are not at the boundary of a substructure are called interior nodes. The associated degrees of freedom are called interior degrees of freedom. Conceptually, a reduced set of equilibrium equations is derived in terms of displacements of the boundary nodes for the entire structure. This is accomplished by eliminating interior degrees of freedom for all substructures from the governing equation. The reduced set of equations is solved for boundary displacements. The interior displacements are then computed from boundary displacements.

Consider one substructure; the equilibrium equation is written as

$$\begin{bmatrix} K_{BB} & K_{IB} \\ K_{BI} & K_{II} \end{bmatrix} \begin{bmatrix} z_B \\ z_I \end{bmatrix} = \begin{bmatrix} F_B \\ F_I \end{bmatrix} \tag{1}$$

where K is the stiffness matrix, F is the load vector, and B and I indicate the boundary and interior quantities, respectively. The degrees of freedom for the interior and boundary nodes can be separated as

$$z_I = -K_{II}^{-1} K_{IB} z_B + K_{II}^{-1} F_I \tag{2}$$

$$z_B = -K_{BB}^{-1} K_{BI} z_I + K_{BB}^{-1} F_B \tag{3}$$

Substituting Eq. (2) into Eq. (3), the interior degrees of freedom are eliminated as

$$\left[K_{BB} - K_{BI} K_{II}^{-1} K_{IB}\right] z_B = F_B - K_{BI} K_{II}^{-1} F_I \tag{4}$$

Note that in Eq. (4) the degrees of freedom z_I are condensed to the boundary degrees of freedom z_B for each substructure that is considered as a superelement. Assembling all superelements, a reduced set of the equilibrium equations is obtained and subsequently solved. Although the size of the reduced set is much smaller than that of the entire structure, it is debatable that the superelement formulation has any advantage over the full structure approach for a one-shot analysis. The reason is that, unlike the full structure analysis, additional matrix operations and a complicated database for bookkeeping are required for the superelement analysis. However, in an iterative design environment as in structural optimization, the overhead is easily offset by the reduction of the problem size. Thus, a significant computer savings can be achieved by this formulation in which only the affected or changed substructures are updated.

For topology optimization problems, two superelements are required. The structure that is kept fixed is defined as the substructure α and the structure that is subjected to change as the residual structure r. Before optimization, the stiffness matrix of the substructure α is formulated and the interior degrees of freedom are transformed to the boundary degrees of freedom that are adjoining the residual structure. The reduced set of the equilibrium equations for the residual structure is written as

$$K_{rr} z_r = F_r \tag{5}$$

During the optimization process, only the residual structure r is solved.

Design Sensitivity Analysis

In addition to the computational advantage over the full structure finite element analysis, the superelement method is more efficient in sensitivity calculations. Consider a full structure; the equilibrium equation is

$$Kz = F \tag{6}$$

Differentiate Eq. (6) with respect to the design variable b, and the sensitivity of the displacement is written as

$$K\frac{\partial z}{\partial b} = -\frac{\partial K}{\partial b} z + \frac{\mathrm{d}F}{\mathrm{d}b} \tag{7}$$